

# GROWTH OF HOMOLOGY TORSION OF METABELIAN GROUPS

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**ABSTRACT.** We study the growth of torsion of the abelianization of finite index subgroups in finitely generated metabelian groups. This complements the results in [3] which covered the finitely presented amenable groups.

## 1. INTRODUCTION

For subgroups  $A, B$  of a group  $G$  we denote by  $[A, B]$  the subgroup of  $G$  generated by all commutators  $[a, b]$  with  $a \in A, b \in B$ . Set  $G' = [G, G]$  and  $G^{ab} = G/G' \simeq H_1(G, \mathbb{Z})$ . Let  $d(G)$  denote the minimal size of a generating set of a group  $G$ . Let  $t(G)$  be the maximal size of a finite subgroup of  $G$  (where we set  $t(G) = \infty$  if there is no such maximum). Note that when  $G$  is a finitely generated abelian group then  $t(G)$  is the size of the torsion subgroup of  $G$  and is always finite.

Given a finitely generated group  $G$  and a subgroup  $H$  of finite index in  $G$  we are interested in the growth of  $t(H^{ab})$  in terms of  $[G : H]$ . This topic is at the crossroads of group theory, operator algebras, geometry and number theory and has intriguing open questions, see [1], [2], [4]. It is easy to see (cf. [1], Lemma 27, restated as Lemma 7 below) that if  $G$  is a finitely presented group then  $t(H^{ab})$  is bounded above by an exponential function in  $[G : H]$ . In case  $G$  is a finitely presented amenable group and  $(H_i)$  is a Farber chain in  $G$  (for example if  $(H_i)$  are normal subgroups of  $G$  with trivial intersection) it is proved in [3] that  $t(H_i^{ab})$  grows subexponentially in  $[G : H_i]$ . By way of contrast [1] also proves that when  $G$  is allowed to vary over all finitely generated solvable groups of derived length 3, then there is no single function  $f$  in terms of  $[G : H_i]$  which bounds  $t(H_i^{ab})$  as  $[G : H_i] \rightarrow \infty$ . One is thus led to the question what happens for finitely generated metabelian groups which are not finitely presented. It is easy to see that some function bounding  $t(H^{ab})$  exists in this class: There are only countably many finitely generated metabelian groups and a diagonal argument produces a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with the following property: Given a finitely generated metabelian group  $G$  there is  $a = a(G) \in \mathbb{N}$  such that if  $H \leq G$  with  $a < [G : H] < \infty$ , then  $t(H^{ab}) < f([G : H])$ . It is natural to expect that this non-constructive bound could be improved when we require that the coset space  $G/H$  approximates  $G$  sufficiently well, for example if  $H$  is member of a chain of normal subgroups  $(H_i)$  with trivial intersection.

Part 1 of Theorem 1 shows that the function  $f$  above can be taken to be any superexponential function, e.g.  $f(n) = n^n$ .

**Theorem 1.** *Let  $G$  be a finitely generated metabelian group and let  $A = G'$ .*

*1. There is a constant  $D$  depending on  $G$  such that  $t(H^{ab}) < D^{[G:H]}$  for every subgroup  $H$  of finite index in  $G$ .*

*2. Let  $(H_i)$  be a sequence of finite index subgroups in  $G$  such that  $[A : (A \cap H_i)] \rightarrow \infty$ . Then*

$$\lim_{i \rightarrow \infty} \frac{\log t(H_i^{ab})}{[G : H_i]} = 0.$$

It is easy to see that the exponential bound in part 1 of Theorem 1 is sharp. Let us take  $G = C_2 \wr \mathbb{Z}$  and for  $n \in \mathbb{N}$  let  $H_n = \pi^{-1}(n\mathbb{Z})$  where  $\pi : G \rightarrow \mathbb{Z}$  is the homomorphism of  $G$  onto the top group  $\mathbb{Z}$ . Then  $t(H_n^{ab}) = 2^n$ .

As a by-product of our method we can give a short proof of a special case of a theorem of Luck from [4].

**Theorem 2.** *Let  $G$  be a finitely presented group with an infinite abelian normal subgroup  $A$ . Let  $(H_i)$  be a sequence of finite index subgroups of  $G$  with  $[A : (A \cap H_i)] \rightarrow \infty$ . Then*

$$\lim_{i \rightarrow \infty} \frac{\log t(H_i^{ab})}{[G : H_i]} = 0.$$

Luck's result proves subexponential growth of the torsion of integral homology in all degrees (provided  $G$  has type  $F$ ) in the more general situation when  $G$  has an infinite normal elementary amenable subgroup but under the stronger assumption that  $(H_i)$  is a normal chain in  $G$  with trivial intersection.

**A question.** Let  $G$  be an amenable group of type  $F_{n+1}$ . Corollary 2 of [3] proved subexponential growth of  $t(H_n(M_i, \mathbb{Z}))$  for any Farber chain of finite index subgroups  $(M_i)$  in  $G$ . In view of Theorem 1 we can ask whether in case  $G$  is a metabelian group the conclusion holds under the weaker assumption that  $G$  is of type  $F_n$  or even  $FP_n$ .

**Question 3.** *Let  $G$  be a metabelian group of type  $F_n$  and let  $(M_i)$  be a chain of normal subgroups with trivial intersection in  $G$ . Is it true that*

$$\lim_{i \rightarrow \infty} \frac{t(H_n(M_i, \mathbb{Z}))}{[G : M_i]} = 0?$$

Note that we need that  $G$  is at least of type  $FP_n$  in order to guarantee that  $H_n(G, \mathbb{Z})$  is finitely generated.

## 2. PROOFS

We begin with some elementary results.

**Proposition 4.** *Let  $N$  be a normal subgroup of a group  $G$ . Then*

$$t(N) \leq t(G) \leq t(N)t(G/N).$$

*If  $G/N$  is torsion-free then  $t(N) = t(G)$ . If  $N$  is finite then  $t(G) = |N|t(G/N)$ .*

*Proof.* This is clear.  $\square$

**Lemma 5.** *Let  $M$  be a right  $\mathbb{Z}[G]$ -module and let  $L \leq M$  be a submodule of finite index. Then  $[M(G-1) : L(G-1)] \leq [M : L]^d$  where  $d = d(G)$*

*Proof.* Let  $g_1, \dots, g_d$  be a generating set for  $G$  of minimal size. Note that  $M(G-1) = \sum_{i=1}^d M(g_i - 1)$ . Therefore the map  $f : M^d \rightarrow M(G-1)$  defined by  $f(m_1, \dots, m_d) = \sum_{i=1}^d m_i(g_i - 1)$  ( $m_i \in M$ ) is surjective. Similarly  $f(L^d) = L(G-1)$  and so  $f$  induces an additive group homomorphism

$$\bar{f} : \left( \frac{M}{L} \right)^d \longrightarrow \frac{M(G-1)}{L(G-1)}$$

which is surjective. The claim of the lemma follows.  $\square$

The following Lemma is well known (compare with Lemma 6 of [3]). For a vector  $v = (x_1, \dots, x_n) \in \mathbb{Z}^n$  we denote by  $|v|$  the  $l_1$ -norm of  $v$ , i.e.  $|v| = \sum_{i=1}^n |x_i|$ .

**Lemma 6.** *Let  $n \in \mathbb{N}$  and let  $v_1, \dots, v_k \in \mathbb{Z}^n$ . Let  $G = \mathbb{Z}^n / (\sum_{j=1}^k \mathbb{Z}v_j)$ . Then  $t(A) \leq \prod_{i=1}^n c_i$  where  $c_1, \dots, c_n$  are the  $n$  largest values from the list  $|v_1|, \dots, |v_k|$ .*

*Proof.* Let  $X$  be the  $n \times k$  matrix with rows  $v_1, \dots, v_k$ . Then  $t(A)$  is the g.c.d of the non-zero minors of maximal rank in  $X$ . Any such minor has rank at most  $n$  and so is at most  $c_1 \cdots c_n$ .  $\square$

**Lemma 7** (Lemma 27 of [1]). *Let  $G$  be a finitely presented group. There is a constant  $C$  depending on  $G$  such that  $t(H^{ab}) \leq C^{[G:H]}$  for any subgroup  $H$  of finite index in  $G$ .*

**Proposition 8.** *Let  $G$  be group with a normal abelian subgroup  $A$ . Let  $H$  be a subgroup of finite index in  $G$  such that  $HA = G$ . Then*

$$t(H^{ab}) \leq [G : H]^{d(G/A)} t(G^{ab}).$$

*Proof.* Let  $B = A \cap H$ , this is a normal subgroup of  $G$  with  $[A : B] = [G : H] = n$  say. Considering  $A$  and  $B$  as  $\mathbb{Z}[G/A]$ -modules (with the action of  $G/A$  on  $A$  by conjugation) Lemma 5 gives  $[[A, G] : [B, G]] \leq n^d$  where  $d = d(G/A)$ .

The group  $A/[A, G]$  is central in  $G/[A, G]$  and expanding  $G' = [G, G] = [HA, HA] \bmod [A, G]$  we obtain  $G' = H'[A, G]$ . Therefore  $[G' : H'] = [[A, G] : ([A, G] \cap H')] \leq [[A, G] : [B, G]]$  since  $B = H \cap A$  and so  $H' \geq [B, H] = [B, G]$ . Therefore  $[G' : H'] \leq n^d$  and in particular  $[(G' \cap H) : H'] \leq n^d$ .

Now we can apply Proposition 4 to  $H/H'$  with a finite normal subgroup  $(G' \cap H)/H'$  and we obtain

$$t(H/H') = |(G' \cap H)/H'| t(H/(G' \cap H)) \leq n^d t(H/(G' \cap H)).$$

On the other hand  $H/(G' \cap H) \simeq HG'/G' \leq G/G'$ . Hence  $t(H/(G' \cap H)) = t(HG'/G') \leq t(G/G')$  and the result follows.  $\square$

*Proof of Theorem 2.* . Let  $G_i = AH_i$  and  $m_i = [G : G_i]$ ,  $a_i = [G_i : H_i] = [A : (A \cap H_i)]$ . Thus  $[G : H_i] = m_i a_i$  with  $a_i \rightarrow \infty$ . Let  $d = d(G)$ . The Nielsen-Schreier theorem gives  $d(G_i) \leq (d-1)m_i + 1 \leq dm_i$ . By Lemma 7 there is a constant  $C$  depending on  $G$  such that  $t(G_i^{ab}) \leq C^{m_i}$ . Now we apply Proposition 8 to  $G_i$  with normal subgroup  $A$  and a finite index subgroup  $H_i$  (which satisfies  $H_i A = G_i$  by the definition of  $G_i$ ). We obtain

$$t(H_i^{ab}) \leq a_i^{d(G_i/A)} t(G_i^{ab}) \leq a_i^{dm_i} C^{m_i}.$$

Therefore

$$(1) \quad \frac{\log t(H_i^{ab})}{[G : H_i]} \leq \frac{m_i(d \log a_i + \log C)}{a_i m_i} = \frac{d \log a_i + \log C}{a_i} \rightarrow 0,$$

since  $a_i^{-1} \log a_i \rightarrow 0$  as  $a_i \rightarrow \infty$ .  $\square$

*Proof of Theorem 1.* We need the following.

**Proposition 9.** *Let  $G$  be a finitely generated metabelian group. There is a constant  $C$  depending on  $G$  with the following property: Let  $H$  be a subgroup of finite index in  $G$  containing  $G'$ . Then  $t(H^{ab}) \leq C^{[G:H]}$ .*

Let us postpone the proof of Proposition 9 for the moment and finish the proof of Theorem 1.

Define  $G_i = AH_i$  and set  $m_i = [G : G_i]$ ,  $a_i = [G_i : H_i] = [A : (A \cap H_i)]$  so that  $[G : H_i] = a_i m_i$ . By Proposition 9 we have  $t(G_i^{ab}) \leq C^{m_i}$  for some constant  $C$  depending only on  $G$ . On the other hand we can apply Proposition 8 to  $G_i$  with a normal abelian subgroup  $A$  and a finite index subgroup  $H_i$  obtaining  $t(H_i) \leq a_i^{d(G_i/A)} t(G_i^{ab})$ . Since  $G/A$  is abelian we deduce that  $d(G_i/A) \leq d(G/A) = d$  say, and so  $t(H_i^{ab}) \leq a_i^d C^{m_i}$ . Part 1 follows since  $a_i^d C^{m_i} \leq 2^{da_i} C^{m_i} \leq (2^d C)^{[G:H_i]}$  and we can take  $D = 2^d C$ .

Part 2 of Theorem 1 easily follows from a computation similar to (1) using that  $a_i \rightarrow \infty$ .  $\square$

It remains to prove Proposition 9.

*Proof of Proposition 9.* Let  $g_1, \dots, g_d$  be a generating set of  $G$ . Let  $n = [G : H]$  and note that  $H$  is normal in  $G$  since  $H \geq G'$ . Let  $L := G^n G'$ , then  $L \leq H$  and  $[G : L] \leq n^d$ . Let  $A := G'$ , then  $A$  is a  $\mathbb{Z}[G^{ab}]$  module generated by  $\{[g_i, g_j] \mid 1 \leq i < j \leq d\}$ . Let  $W := [A, H] \leq A$ . The quotient  $H/W$  is a finitely generated nilpotent group of class at most 2 and  $LW/W$  is a subgroup of finite index in  $H/W$ . Therefore  $(LW/W)' = L'W/W$  has finite index in  $(H/W)' = H'W$ , i.e. the index  $[H' : L'W]$  is finite. By Proposition

4  $t(H/L'W) = t(H/H')|H'/LW|$  and in particular  $t(H/H') \leq t(H/L'W)$ . In turn  $t(H/L'W) \leq t(H/A)t(A/L'W) \leq t(G^{ab})t(A/L'W)$ . We will find a bound for  $t(A/L'W)$ . Note that  $A/W$  is in the centre of  $L/W$  and  $L = \langle g_1^n, \dots, g_d^n \rangle A$ . Hence  $L'W = \langle [g_i^n, g_j^n] \mid 1 \leq i < j \leq d \rangle W$ .

Let

$$X := \bigoplus_{1 \leq i < j \leq d} e_{i,j} \mathbb{Z}[G^{ab}]$$

be the free  $\mathbb{Z}[G^{ab}]$ -module with basis  $\{e_{i,j} \mid 1 \leq i < j \leq d\}$  and let  $f : X \rightarrow A$  be the surjective  $\mathbb{Z}[G^{ab}]$ -module homomorphism such that  $f(e_{i,j}) = [g_i, g_j]$ . Since  $X$  is a Noetherian module  $\ker f$  is generated (as a  $\mathbb{Z}[G^{ab}]$ -module) by finitely many elements, say  $\{r_1, \dots, r_k\} \subset X$ .

Let  $Y := X/X(H-1) = \bigoplus_{i < j} e_{i,j} \mathbb{Z}[G/H]$  and denote by  $\pi : X \rightarrow Y$  the natural quotient map such that  $\pi(e_{i,j} \cdot aG') = e_{i,j} \cdot aH$  for each  $e_{i,j}$  and each  $a \in G$ . Again there is a unique  $\mathbb{Z}[G/H]$ -module homomorphism  $h : Y \rightarrow A/W = A/[A, H]$  such that  $h(e_{i,j}) = [g_i, g_j]W$ . We have  $f = h \circ \pi$ .

Using the commutator identities  $[ab, c] = [a, c]^b [b, c]$  and  $[c, ab] = [c, b][c, a]^b$  we can write  $[g_i^n, g_j^n] = \prod_{s=1}^{n^2} [g_i, g_j]^{z_{i,j,s}}$  for some elements  $z_{i,j,s} \in G$ . Define  $u_{i,j} = e_{i,j} \cdot \sum_{s=1}^{n^2} z_{i,j,s} G'$ , we have  $f(u_{i,j}) = [g_i^n, g_j^n]$ .

We have

$$L'W = \langle [g_i^n, g_j^n] \mid 1 \leq i < j \leq d \rangle W = f(X(H-1)) + f\left(\sum_{1 \leq i < j \leq n} \mathbb{Z}u_{i,j}\right)$$

and from  $f = h \circ \pi$  it follows that

$$\frac{A}{L'W} \simeq \frac{Y}{\pi(\ker f) + \sum_{i < j} \mathbb{Z}\pi(u_{i,j})}.$$

We consider the  $l^1$  norm  $|\cdot|_X$  on the free  $\mathbb{Z}$ -module  $X$  (respectively the  $l^1$  norm  $|\cdot|_Y$  on  $Y$ ) as the sum of the absolute values of coordinates computed with respect to the standard  $\mathbb{Z}$ -basis  $\{e_{i,j}\bar{g} \mid i < j, \bar{g} \in G^{ab}\}$  of  $X$  (respectively the  $\mathbb{Z}$ -basis  $\{e_{i,j}\tilde{g} \mid i < j, \tilde{g} \in G/H\}$  of  $Y$ ). Note that for any element  $v \in X$  we have  $|v|_X \geq |\pi(v)|_Y$  and also  $|v|_X = |v \cdot \bar{g}|_X$  for any  $\bar{g} \in G^{ab}$ .

Let  $c = \max\{|r_1|_X, \dots, |r_k|_X\}$  where  $r_1, \dots, r_k$  are the  $\mathbb{Z}[G^{ab}]$  module generators of  $\ker f$ . Then  $\pi(\ker f)$  is generated as  $\mathbb{Z}$ -module by the set  $T := \{\pi(r_s)\tilde{g} \mid s = 1, \dots, k, \tilde{g} \in G/H\}$ . Observe that  $|\pi(u_{i,j})|_Y \leq |u_{i,j}|_X \leq n^2$  for each  $1 \leq i < j \leq d$  while for each  $w = \pi(r_s)\tilde{g}$  we have  $|w|_Y \leq |r_s|_X \leq c$ .

The group  $(Y, +)$  is a free  $\mathbb{Z}$ -module of rank  $nd(d-1)/2$  and  $A/L'W \simeq Y/Z$  where

$$Z = \pi(\ker f) + \sum_{1 \leq i < j \leq d} \mathbb{Z}\pi(u_{i,j}) = \sum_{w \in T} \mathbb{Z}w + \sum_{1 \leq i < j \leq d} \mathbb{Z}\pi(u_{i,j}).$$

Lemma 6 applied to  $Y/Z$  gives  $t(A/L'W) \leq (n^2)^{d(d-1)/2} c^{nd(d-1)/2}$  and so

$$t(H/H') \leq t(G^{ab})t(A/L'W) \leq t(G^{ab})(n^2)^{d(d-1)/2} c^{nd(d-1)/2}$$

Since  $t(G^{ab})$  does not depend on  $H$  and  $n^2 < 3^n$  we may take  $C = (t(G^{ab})3c)^{d^2/2}$  giving  $t(H/H') < C^n$  as required.  $\square$

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